Videns piscator a militibus se comprehendi putavit occidi. Ingressus Appollonius jussit eum adduci ad se et ait: "hic est paranymphus meus, qui mihi post naufragium opem dedit et ad civitatem venire ostendit." Et dicit ei: "ego sum Tyrius Appollonius." Et jussit sibi dari CC sistercias argenti, servos et ancillas, et fecit eum comitem suum, quamdiu vixit. Elamitus vero, qui ei de Antiocho nunciavit, procidens ad pedes Appollonii . . . . et ait: "domine, memor esto Elamiti servi tui!" Appollonius apprehensa manu eius erexit eum fecitque eum divitem et ordinavit comitem. Hiis expletis genuit Appollonius filium de conjuge sua, quem in loco avi sui Altistratis constituit regem. Vixit vero Appollonius cum conjuge sua annos LXXIV et tenuit regnum Antioche et Tyri et Tyrenensium quiete ac feliciter. Casus suos ipse descripsit, ipse duo volumina perfecit, unum in templo Ephesiorum, alterum in sua bibliotheca collocavit. Et defunctus est et perrexit ad vitam eternam, ad quam vitam nos perducat, qui sine fine vivit et regnat Amen.

ON THE QUATERNION GROUP.

BY G. A. MILLER, PH.D.

(*Read October 7, 1898.*)

Although the quaternion group (Q) has received some attention,[1] yet many of the properties of this important group remain to be investigated. It is the object of this paper to enter upon the study of some of these group properties after stating the known principles which underlie the investigations that follow. We shall also determine the different ways in which Q may be represented as a substitution group.

It is well known that every group of a finite order may be represented as a regular substitution group and that any two regular substitution groups which are simply isomorphic are also conjugate.

A complete list of the regular substitution groups of order g must therefore include every possible group of this order and no group can occur twice in such a list. In following Prof. Cayley's

[1] Dedekind, *Mathematische Annalen*, 1897, Vol. xlviii, pp. 549–552.

notation we represent Q as a regular substitution group in the following manner: [1]

1	*ae. bf. cg. dh*	*aceg. bdfh*
		agec. bhfd
		abef. chgd
		afeb. cdgh
		adeh. bgfc
		ahed. bcfg

The Different Ways in Which Q May be Represented as a Substitution Group.

We observe, in the first place, that Q cannot be represented as a non-regular transitive substitution group. If such a representation were possible Q would have to contain some subgroup of a prime order that is not self-conjugate.[2] As it contains only one subgroup of order 2 this must clearly be self-conjugate. Hence we observe that *there is only one transitive substitution group that is simply isomorphic to Q.*

It is known that the number of the intransitive substitution groups that are simply isomorphic to a given group is an increasing function of the degree, which becomes infinite when the degree becomes infinite. We proceed to determine the nature of this function in the present case. Since every group whose order is the square of a prime number is Abelian, a substitution group which is simply isomorphic to Q must contain at least one transitive constituent of order 8 and its degree must be $2n$, n being a positive integer greater than 3.

We have seen that Q contains only one subgroup of order 2. With respect to this it is isomorphic to the four-group, since this subgroup contains the square of each one of its operators. As a subgroup whose order is one-half of the order of the entire group must always be self-conjugate, Q contains three self-conjugate subgroups of order 4. Since none of these three subgroups is characteristic[3] they must be transformed into each other by the largest

[1] Cayley, *Quarterly Journal of Mathematics*, 1891, Vol. xxv, p. 144.

[2] Cf. Dyck, *Mathematische Annalen*, 1883, Vol. xxii, p. 90. It may be remarked that the statement on p. 101 of this article that a group which can be represented only in the regular form contains only self-conjugate subgroups is not quite correct, as may also be inferred from other parts of the same article.

[3] Frobenius, *Berliner Sitzungsberichte*, 1895, p. 183.

group that contains Q as a self-conjugate subgroup. Hence we need to consider only one of these three subgroups in connection with the study of the intransitive substitution groups that are simply isomorphic to Q.

We may now state the problem of finding all the substitution groups that are simply isomorphic to Q in the following manner. Such a group contains a transitive constituents of order 8, where a is an integer greater than 0. Its other constituents form a group whose order is either 4 or 2. If this order is four these constituents must form the four-group. If it is two these can form only one group for a given set of values of n and a. Hence we observe that the number of quaternion substitution groups of degree $2n$, $n > 3$, which contain no constituent group of order 4 is a_2, where a_2 is the largest integral value of x that satisfies the relation:

$$x \leqq \frac{n}{4}.$$

To find the number of these groups that contain a constituent of order 4 we may first find the number of those that contain only one transitive constituent of order 8, then the number of those that contain two such constituents, etc. The sum of these numbers is the number required. Each of these numbers may be directly found by means of the following formula,[1] in which N is the number of all the possible substitution groups of order 4 and degree $2n$, m is any positive integer, and a_1 is the largest value of y that satisfies the relation

$$y \leqq \frac{n}{2}$$

When $n = 6m$, $\quad N = m(3m^2 + 6m + 1) + a_1$

" $n = 6m + 1$, $\quad N = \frac{m(6m^2 + 15m + 5)}{2} + a_1$

" $n = 6m + 2$, $\quad N = 3m(m+1)(m+2) + 1 + a_1$

" $n = 6m + 3$, $\quad N = \frac{(2m+1)(3m^2 + 9m + 4)}{2} + a_1$

" $n = 6m + 4$, $\quad N = (m+1)(3m^2 + 9m + 4) + a_1$

" $n = 6m + 5$, $\quad N = \frac{3(m+1)(2m^2 + 7m + 4)}{2} + a_1$

[1] Miller, *Philosophical Magazine*, 1896, Vol. xli, p. 437.

If we add a_2 to the sum of the numbers obtained by means of these formulas we obtain the total number of the substitution groups of degree $2n$ that are simply isomorphic to Q. Among these substitution groups the given regular group is especially convenient for the study of the properties of Q.

In what follows we shall, therefore, suppose Q written in this way unless the contrary is explicitly stated.

It is known that all the substitutions that involve no more than g letters and are commutative to every substitution of a regular group involving the same g letters form a group which is conjugate to the regular group.[1] This conjugate of the given regular group contains the following substitutions:

1	$ae.\ bf.\ cg.\ dh$	$aceg.\ bhfd$
		$agec.\ bdfh$
		$abef.\ cdgh$
		$afeb.\ chgd$
		$adeh.\ bcfg$
		$ahed.\ bgfc$

One of the 192 substitutions in these 8 letters that transform one of these two regular groups into the other is the transposition dh.

The Group of Isomorphisms of Q.

The largest group in these eight letters that transforms one of the two given regular groups into itself must be transitive, since it includes a regular group. Its subgroup which includes all its substitutions that do not involve a given letter is the group of isomorphisms of Q. We proceed to prove that this is simply isomorphic to the symmetric group of order 24. To prove this we observe that an operator of order 4 may be made to correspond to any other operator of this order in a simple isomorphism of Q to itself. Hence the first correspondence can be effected in 6 ways and the second can evidently be effected in 4 ways, so that the group of isomorphisms must be of order 24.

This group of isomorphisms may be represented as a transitive substitution group of degree 6, since there are 6 operators of order

[1] Jordan, *Traité des Substitutions*, p. 60.

4 that can be made to correspond and these generate *Q*. As this substitution group cannot contain a substitution whose degree is less than 4 and the transitive groups of degree 6 and order 24 that have this property are simply isomorphic to the symmetric group of this order it follows directly that *the group of isomorphisms of Q is the symmetric group of order* 24 *and that the group of cogredient isomorphisms is its self-conjugate subgroup of order* 4.

There are two transitive groups of degree 6 that are simply isomorphic to the symmetric group of order 24. In one of these the subgroup which contains all the substitutions that do not include a given element is the cyclical group of order 4 while in the other it is the four-group. It remains to determine which of these two groups is the substitution group of isomorphisms of *Q*. This may be easily done by making *Q* simply isomorphic to itself in the following manner:

1	1	*abef. chgd*	*adeh. bgfc*
ae. bf. cg. dh	*ae. bf. cg. dh*	*afeb. cdgh*	*ahed. bcfg*
aceg. bdfh	*aceg. bdfh*	*adeh. bgfc*	*afeb. cdgh*
agec. bhfd	*agec. bhfd*	*ahed. bcfg*	*abef. chgd*

The substitution which corresponds to this isomorphism is given by the second columns of letters; hence it is bdfh and the substitution group of isomorphisms of *Q* is the one which Prof. Cayley represents by $(\pm \text{abcdef})_{24}$.[1]

It is known that *Q* is simply isomorphic to the eight unities (1, —1, i, —i, j, —j, k, —k) of the *quaternion number system*. As *Q* can be made simply isomorphic to itself in 24 different ways the simple isomorphism of *Q* to these unities or of these unities to themselves may also be written in 24 ways. The following is one of these ways:

1	1	*abef. chgd*	*j*
ae. bf. cg. dh	—1	*afeb. cdgh*	—*j*
aceg. bdfh	*i*	*adeh. bgfc*	—*k*
agec. bhfd	—*i*	*ahed. bcfg*	*k*

It may be very easily verified that the following relations are

[1] *Quarterly Journal of Mathematics*, 1891, Vol. xxv, p. 80.

satisfied by the substitutions which correspond to the unities that are employed.[1]

$$
\begin{array}{lll}
ij = k & ji = -k & i^2 = -1 \\
jk = i & kj = -i & j^2 = -1 \\
ki = j & ik = -j & k^2 = -1
\end{array}
$$

These relations between the quaternion unities could also have been obtained directly by means of the corresponding substitutions.

As any relation between quaternion unities remains true if we replace all these unities by those which correspond to them in any simple isomorphism of their group to itself, it follows directly that a knowledge of the group of isomorphisms of this group to itself is of great utility in transforming quaternion relations; *e. g.*, from the simple isomorphism

$$
\begin{array}{r|l}
1 & 1 \\
-1 & -1 \\
i & j \\
-i & -j
\end{array}
\qquad\qquad
\begin{array}{r|l}
j & k \\
-j & -k \\
-k & -i \\
k & i
\end{array}
$$

it follows that i may be replaced by j, j by k, and k by i at the same time. In other words, we may always perform the substitution $ijk.\ (-i)\ (-j)\ (-k)$ on the three imaginary unities of quaternions. By means of this substitution we can obtain each of the three relations given above from any one of the set. The twenty-four possible substitutions in these imaginary unities can be directly obtained from the given group of isomorphisms of Q. They are the following:

	$ijk.\ (-i)\ (-j)\ (-k)$	$j\ (-k)\ (-j)\ k$	$ij.\ (-i)\ (-j).\ k\ (-k)$
$-i).\ j\ (-j)$	$i\ (-j)\ (-k).\ jk\ (-i)$	$jk\ (-j)\ (-k)$	$i\ (-i).\ jk.\ (-j)\ (-k)$
$-i).\ k\ (-k)$	$ij\ (-k).\ k\ (-i)\ (-j)$	$ik\ (-i)\ (-k)$	$i\ (-i).\ j\ (-k).\ (-j)\ k$
$-j).\ k\ (-k)$	$i\ (-j)\ k.\ j\ (-k)\ (-i)$	$i\ (-k)\ (-i)\ k$	$i\ (-j).\ j\ (-i).\ k\ (-k)$
	$ikj.\ (-i)\ (-k)\ (-j)$	$i\ (-j)\ (-i)\ j$	$ik.\ j\ (-j).\ (-i)\ (-k)$
	$ik\ (-j).\ j\ (-i)\ (-k)$	$ij\ (-i)\ (-j)$	$i\ (-k).\ j\ (-j).\ k\ (-i)$
	$i\ (-k)\ (-j).\ j\ (-i)\ k$		
	$i\ (-k)\ j.\ k\ (-j)\ (-i)$		

when an equation between the quaternion unities admits α of these

[1] Cf. Tait's *Quaternion*, 1890, p. 46.

substitutions these substitutions must form a subgroup of this group of isomorphism and the given equation must assume $24 \div a$ different forms which are equally true in case it is transformed by all these substitutions, *e. g.*, each of the three equations in the last set given above admits a cyclical subgroup of order 4. Hence each of these equations gives rise to $24 \div 4 = 6$ true equations. In addition to the three that have been given we have $(-i)^2 = (-j)^2 = (-k)^2 = -1$.

We have already noticed that the group of cogredient isomorphisms of Q is the four-group. Hence Q has only two operators that are commutative to each one of its operators. These are evidently the operators which correspond to 1 and -1 in the quaternion unities. These two unities are therefore the only ones in the quarternion number system that are commutative to all the numbers of the system. It need scarcely be remarked that any one of the three cyclical subgroups of order 4 contained in Q may correspond to the unities of the ordinary complex number system.

Relation Between the Quaternion Group and the Hamiltonian Groups.

One of the most remarkable properties of the quaternion group is that each of its subgroups is self-conjugate. Dedekind has called all the groups which have this property *Hamiltonian groups* and he has pointed out that the quaternion group is of fundamental importance in the study of the Hamiltonian groups.[1] It has recently been proved that every Hamiltonian group is the direct product of an Abelian group of an odd order and a Hamiltonian group of order 2^a, and that there is one and only one Hamiltonian group of order 2^a for every integer value of a greater than 2.[2]

It is easy to see that the direct product of the quaternion group and the Abelian group of order 2^{a-3} which contains $2^{a-3} - 1$ operators of order 2 is Hamiltonian. Since there is only one Hamiltonian group of this order it follows that every such Hamiltonian group may be constructed in this manner. Hence we have that *every Hamiltonian group whose order is divisible by* 2^a, *but not* 2^{a+1} *must be the direct product of some Abelian group of an odd order, the Abelian group of order* 2^{a-3} *which contains* $2^{a-3} - 1$ *operators of order* 2, *and the quaternion group.*

[1] Dedekind, *loc. cit.*

[2] Miller, *Comptes Rendus*, 1898, Vol. cxxvi, p. 1406.

While the direct product of the quaternion group and any Abelian group of an odd order is always a Hamiltonian group, the direct product of the quaternion group and an Abelian group whose order is divisible by a power of 2 is only Hamiltonian when the latter group contains no operator whose order is divisible 4. This follows directly from the fact that the group generated by the product of an operator of order 4 in the Hamiltonian group and any operator in such an Abelian group must be self-conjugate.

We may determine the number of the quaternion groups that are contained in a Hamiltonian group whose order is divisible by 2^{α} without being divisible by $2^{\alpha+1}$ in the following manner. Such a group contains a single subgroup[1] of order 2^{α}. This subgroup includes 3 times $2^{\alpha-2}$ operators of order 4. Each quaternion subgroup includes two of the operators of order 4 that are included in a subgroup of order $2^{\alpha-1}$ which involves only $2^{\alpha-2}$ operators of order 4. Hence *there are* $2^{2\alpha\ \ 6}$ *quaternion subgroups in the given Hamiltonian group*. All of these have the commutator subgroup of the entire group in common. In other words, the commutator subgroup of a Hamiltonian group is the same as that of any one of its quaternion subgroups.

CORNELL UNIVERSITY, June, 1898.

Stated Meeting, October 21, 1898.

Vice-President SELLERS in the Chair.

Present, 12 members.

Prof. Lighter Witmer, a newly elected member, was presented to the Chair, and took his seat.

The minutes of the last stated meeting were read and approved.

Dr. Frazer read a letter from the International Geological Congress in regard to the establishment of an international floating institute, and offered the following resolution:

Resolved, That the President of the Society be requested to memorialize Congress in favor of an appropriation in aid of the in-

[1] Sylow, *Mathematische Annalen*, 1872, Vol. v, p. 584.